TRINITY COLLEGE

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

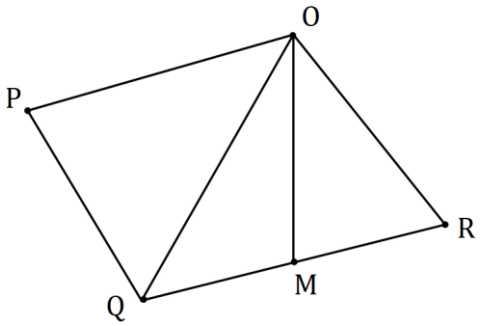
Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(5 marks)**

In the diagram below, M is the midpoint of QR.



If $\overrightarrow{OP} = \mathbf{p}, \overrightarrow{OQ} = \mathbf{q}$ and $\overrightarrow{OR} = \mathbf{r}$, express the following in terms of $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$.

(a) $\overrightarrow{QP}$. (1 mark)

Solution
$\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$
Specific behaviours
✓ correct expression

(b) $\overrightarrow{OM}$. (2 marks)

Solution
$\overrightarrow{OM} = \overrightarrow{OR} + \frac{1}{2}\overrightarrow{RQ}$ $= \mathbf{r} + \frac{1}{2}(\mathbf{q} - \mathbf{r})$ $= \frac{1}{2}\mathbf{q} + \frac{1}{2}\mathbf{r}$
Specific behaviours
✓ indicates correct method ✓ correct expression

(c) $8\overrightarrow{MP}$. (2 marks)

Solution
$\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP} = \mathbf{p} - \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{r}$ $8\overrightarrow{MP} = 8\mathbf{p} - 4\mathbf{q} - 4\mathbf{r}$
Specific behaviours
✓ indicates $\overrightarrow{MP}$ ✓ correct expression

See next page

Question 10 **(8 marks)**

Points P, Q and R have coordinates $(-2, 11), (8, 15)$ and $(17, 3)$ respectively. Determine

(a) $\overrightarrow{PQ}$. (1 mark)

Solution
$\overrightarrow{PQ} = (8, 15) - (-2, 11)$ $= (10, 4)$
Specific behaviours
✓ correct vector

(b) $|\overrightarrow{QR}|$. (2 marks)

Solution
$\overrightarrow{QR} = (17, 3) - (8,15)$ $= (9, -12)$ $\left\|\overrightarrow{QR}\right\| = 15$
Specific behaviours
✓ correct vector ✓ magnitude

(c) $2\overrightarrow{PQ} - 60\mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction $\overrightarrow{QR}$. (3 marks)

Solution
$\mathbf{u} = \frac{1}{15}(9, -12)$ $2\overrightarrow{PQ} - 60\mathbf{u} = 2(10, 4) - \frac{60}{15}(9, -12)$ $= (-16, 56)$
Specific behaviours
✓ indicates unit vector ✓ expression for result ✓ correct vector

(d) The coordinates of point S, given that $\overrightarrow{RS} = \overrightarrow{QP}$. (2 marks)

Solution
$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{RS}$ $= \overrightarrow{OR} - \overrightarrow{PQ}$ $= (21, 3) - (10, 4)$ $= (7, -1)$
Specific behaviours
✓ expression for result ✓ correct coordinates

See next page

Question 11 **(8 marks)**

(a) Show that the vectors $(-10, 5)$ and $(4.5, 9)$ are perpendicular. (2 marks)

Solution
$\begin{pmatrix}-10\\5\end{pmatrix} \cdot \begin{pmatrix}4.5\\9\end{pmatrix} = 45 - 45 = 0$ Hence perpendicular as scalar (dot) product is 0.
Specific behaviours
✓ uses dot product ✓ explains result

(b) Determine, to the nearest degree, the angle between the vectors $(1, -5)$ and $(2, -3)$. (2 marks)

Solution
Using CAS: $\theta = 22.38 \approx 22°$ Or: $\theta = \cos^{-1}\left(\frac{17}{\sqrt{26}\times\sqrt{13}}\right)$
Specific behaviours
✓ indicates method ✓ correct angle

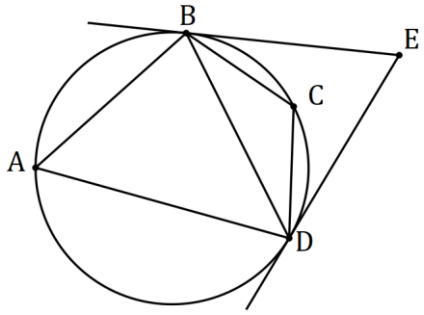
(c) The vectors $(a, 2a - 1)$ and $(a - 4, 3)$ are perpendicular, where a is a constant. Determine the value(s) of a and the corresponding pair(s) of vectors. (4 marks)

Solution
$\begin{pmatrix}a\\2a-1\end{pmatrix} \cdot \begin{pmatrix}a-4\\3\end{pmatrix} = a^2 - 4a + 6a - 3 = 0$ $(a + 3)(a - 1) = 0 \Rightarrow a = -3, a = 1$ $a = -3 \Rightarrow \begin{pmatrix}-3\\-7\end{pmatrix}$ and $\begin{pmatrix}-7\\3\end{pmatrix}$ $a = 1 \Rightarrow \begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-3\\3\end{pmatrix}$
Specific behaviours
✓ uses dot product to form equation ✓ solves equation ✓ states one pair of vectors ✓ states both pairs of vectors

See next page

Question 12 **(7 marks)**

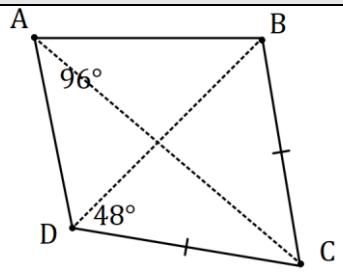
(a) In the diagram below (not drawn to scale) A, B, C and D lie on a circle and EB and ED are tangents to the circle. If $\angle BED = 54°$ and $\angle CDB = 20°$, determine the size of $\angle CBD$. (3 marks)



Solution
$\angle BDE = (180 - 54) \div 2 = 63$ $\angle CDE = 63 - 20 = 43$ $\angle CBD = \angle CDE = 43°$ (AltSegment)
Specific behaviours
✓ $\angle BDE$ ✓ $\angle CDE$ ✓ $\angle CBD$

(b) Quadrilateral $ABCD$ is such that $CB = CD$, $\angle BAD = 96°$ and $\angle BDC = 48°$.

(i) Sketch a diagram to show this information. (1 mark)

Solution

Specific behaviours
✓ correct diagram

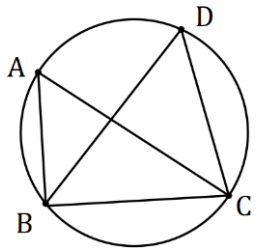
(ii) Show that $ABCD$ is cyclic and hence determine the size of $\angle CAD$. (3 marks)

Solution
$\angle CBD = \angle CDB = 48$ $\angle BCD = 180 - 2 \times 48 = 84$ $\angle BAD + \angle BCD = 96 + 84 = 180$ Hence cyclic as opposite angles supplementary. $\angle CAD = \angle CBD = 48°$ (Same arc)
Specific behaviours
✓ use isosceles triangle for $\angle BCD$ ✓ uses supplementary angles for cyclic ✓ correct size of $\angle CAD$

See next page

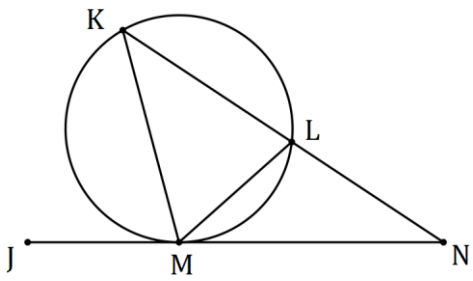
Question 13 **(8 marks)**

(a) A, B, C and D lie on a circle with diameter AC (diagram not to scale). Determine the size of $\angle BDC$ when $\angle BCA = 40°$. (2 marks)



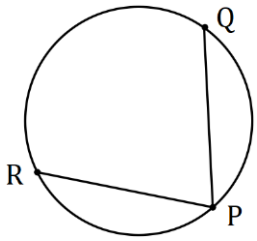
Solution
$\angle BAC = 90 - 40 = 50$ $\angle BAD = \angle BAC = 50°$
Specific behaviours
✓ uses angle in semi-circle for $\angle BAC$ ✓ correct value

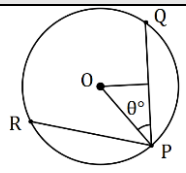
(b) K, L and M lie on a circle (diagram not to scale). Secant KN cuts the circle at L and JN is a tangent to the circle at M. Given that $\angle LNM = 33°$ and $\angle LMN = 43°$, determine the size of $\angle MKL$ and the size of $\angle KMJ$. (3 marks)



Solution
$\angle MKL = \angle LMN = 43°$ (Alternate segments) $\angle KLM = 33 + 43 = 76$ (Exterior angle) $\angle KMJ = \angle KLM = 76°$ (Alternate segments)
Specific behaviours
✓ $\angle MKL$ ✓ $\angle KLM$ ✓ $\angle KMJ$

(c) P, Q and R lie on a circle of radius 54 mm (diagram not to scale) and $PQ = PR = 68$ mm. Determine the size of angle $\angle QPR$, to the nearest degree. (3 marks)



Solution
 $68 \div 2 = 34$ $\theta = \cos^{-1}\frac{34}{54} = 51.0°$ $\angle QPR = 2\theta = 102°$
Specific behaviours
✓ completes diagram ✓ uses trig ratio for half-angle ✓ correct angle

See next page

Question 14 **(9 marks)**

The parts of this question refer to the word DECOMPOSITION. It has 7 different consonants and 6 vowels, some of which are repeated.

(a) Determine the number of ways that 4 different consonants chosen from the letters of the word can be arranged in a row. (1 mark)

Solution
${}^7P_4 = 840$
Specific behaviours
✓ correct number

(b) Determine the number of ways that all the letters of the word can be arranged in a row. (2 marks)

Solution
$\frac{13!}{3! \times 2!} = 518\,918\,400$
Specific behaviours
✓ attempts to account for repeated letters ✓ correct number

(c) Determine the number of ways that all the letters of the word can be arranged in a row if the consonants must all be adjacent. (3 marks)

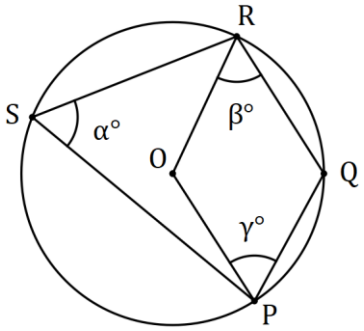
Solution
$\frac{(6+1)! \times 7!}{3! \times 2!} = 2\,116\,800$
Specific behaviours
✓ counts consonants as single group ✓ counts ways to arrange consonants ✓ correct number

(d) Determine how many 3 letter permutations (e.g. MTE, ONO, etc) can be made using the letters of the word. (3 marks)

Solution
All different: $10 \times 9 \times 8 = 720$ Two I's and one other: $3 \times 9 = 27$ Two O's and one other: $3 \times 9 = 27$ Three O's: 1 Total: $720 + 27 + 27 + 1 = 775$
Specific behaviours
✓ attempts to consider separate cases ✓ correct number containing 2 I's and 2 O's ✓ correct total

Question 15 **(8 marks)**

(a) In the diagram below (not drawn to scale) P, Q, R and S lie on the circle with centre O. Determine the size of angles α, β and γ given that $\angle PQR = 128°$ and $5\beta = 3\gamma$. (4 marks)

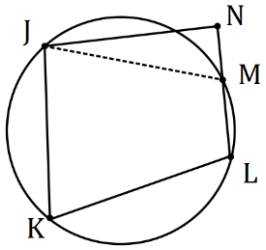


Solution
$\alpha = 180 - 128 = 52°$ $\beta + \gamma = 128$ $3\beta + 3\gamma = 384 \Rightarrow 8\beta = 384 \Rightarrow \beta = 48°$ $\gamma = 80°$
Specific behaviours
✓ correct α ✓ equation for $\beta + \gamma$ ✓ correct γ ✓ correct β

(b) Write the converse of the theorem that states the opposite angles of a cyclic quadrilateral are supplementary. (1 mark)

Solution
When opposite angles in a quadrilateral are supplementary, the quadrilateral is cyclic.
Specific behaviours
✓ correct statement

(c) Prove by contradiction that the converse you wrote in (b) is true. Start by assuming that there is a quadrilateral that *does* have supplementary opposite angles but is *not* cyclic, such as $JKLN$ shown below. (3 marks)



Solution
From assumption, $\angle N = 180° - \angle K$. But from regular theorem, $\angle M = 180° - \angle K$. Hence $\angle N = \angle M$, but this is impossible (as JN and JM would then be parallel and triangle JMN would not exist). Thus, our original assumption must be wrong, and the converse must be true.
Specific behaviours
✓ uses assumption ✓ develops contradiction ✓ explains contradiction and makes deduction

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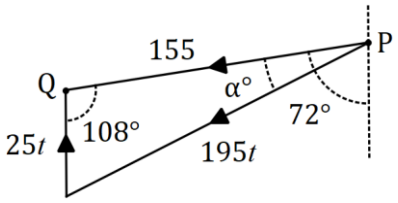
Question 16 **(7 marks)**

Airport P lies 155 km away from airport Q on a bearing of $072°$. A plane leaves airport P at $9{:}30$ am to fly to airport Q. The plane can maintain a speed of 195 kmh^{-1} and there is a steady wind of 25 kmh^{-1} blowing from the south.

Determine the bearing that the helicopter should steer and the time of its arrival at airport Q, to the nearest minute.

Solution



$$\frac{\sin 108}{195t} = \frac{\sin \alpha}{25t}$$

$$\alpha = 7.0°$$

$$\text{Bearing: } 180 + 72 - 7 = 245°$$

$$180 - 108 - 7 = 65$$

$$\frac{\sin 108}{195t} = \frac{\sin 65}{155}$$

$$t = 0.834 \text{ h}$$
$$= 50 \text{ m}$$

$$\text{Arrive at } 10{:}20 \text{ am}$$

Specific behaviours

- ✓ diagram showing vectors and resultant
- ✓ equation using sin rule for α
- ✓ value of α
- ✓ correct bearing
- ✓ equation using sin rule for t
- ✓ value of t
- ✓ correct arrival time

See next page

Question 17 **(7 marks)**

Three forces $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ act on a point in a plane.

The forces are $\mathbf{a} = -44\mathbf{i} + 66\mathbf{j}$ N, $\mathbf{b} = -12\mathbf{i} - 75\mathbf{j}$ N and $\mathbf{c} = 180\mathbf{i} + 102\mathbf{j}$ N.

(a) Determine the magnitude of the resultant force and the direction, to the nearest degree, that the resultant makes with the vector $\mathbf{i}$. (3 marks)

Solution
$\mathbf{r} = \begin{pmatrix}-44\\66\end{pmatrix} + \begin{pmatrix}-12\\-75\end{pmatrix} + \begin{pmatrix}180\\102\end{pmatrix} = \begin{pmatrix}124\\93\end{pmatrix}$ $\lvert\mathbf{r}\rvert = 155$ N $\angle = 36.9 \approx 37°$
Specific behaviours
✓ resultant ✓ correct magnitude ✓ correct angle

When $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = 0$, the forces are in equilibrium.

(b) Determine the values of the scalar constants λ and μ for equilibrium to occur. (4 marks)

Solution
$\lambda\begin{pmatrix}-44\\66\end{pmatrix} + \mu\begin{pmatrix}-12\\-75\end{pmatrix} + \begin{pmatrix}180\\102\end{pmatrix} = 0$ $-44\lambda - 12\mu + 180 = 0$ $66\lambda - 75\mu + 102 = 0$ $\lambda = 3, \quad \mu = 4$
Specific behaviours
✓ equation using $\mathbf{i}$-coefficients ✓ equation using $\mathbf{j}$-coefficients ✓ solves for λ ✓ solves for μ

See next page

Question 18 **(8 marks)**

(a) A set of cards is numbered with all the odd numbers between 101 and 999. Determine the minimum number of cards that must be selected to ensure that at least 6 cards in the selection have the same last digit. Justify your answer using the pigeonhole principle.

(3 marks)

Solution
Let pigeonholes be digits $1, 3, 5, 7, 9$ and pigeons be the last digit of number on card. Then fill all pigeonholes with 5 pigeons, a total of 25 pigeons. The next pigeon will fill one of the pigeonholes with 6 pigeons, and so the minimum number is 26.
Specific behaviours
✓ defines pigeons and pigeonholes ✓ clear explanation ✓ correct number

(b) Nine different books sit on a shelf, one of which is non-fiction and the rest fiction. A student is told they can take away as many of them as they like but must not leave empty handed. Determine how many different selections can be made

(i) of exactly 6 books. (1 mark)

Solution
$\binom{9}{6} = 84$
Specific behaviours
✓ correct number

(ii) altogether. (2 marks)

Solution
Choose either $1, 2, \ldots$ up to all 9 books: $\sum_{n=1}^{9} \binom{9}{n} = 2^9 - 1 = 511$
Specific behaviours
✓ uses property of Pascals triangle ✓ correct number

(iii) that include the non-fiction. (2 marks)

Solution
Choose non-fiction and then $0, 1, \ldots$ up to 8 others: $\binom{1}{1} \times \sum_{n=0}^{8} \binom{8}{n} = 2^8 = 256$
Specific behaviours
✓ indicates method ✓ correct number

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Question 19 **(8 marks)**

Determine how many of the integers between 1 and 500 inclusive are

(a) divisible by 7. (1 mark)

Solution
$\lfloor 500 \div 7 \rfloor = 71$ $n = 71$
Specific behaviours
✓ correct number

(b) divisible by 7 or 9. (3 marks)

Solution
LCM: $(7,9) = 63$; $\lfloor 500 \div 9 \rfloor = 55$ $\lfloor 500 \div 63 \rfloor = 7$ $n = 71 + 55 - 7 = 119$
Specific behaviours
✓ number divisible by 63 ✓ indicates use of inclusion-exclusion ✓ correct number

(c) divisible by 7 or 9 but not both. (1 mark)

Solution
$n = 119 - 7 = 112$
Specific behaviours
✓ correct number

(d) divisible by 7 or 9 but not 6. (3 marks)

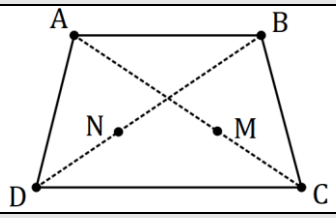
Solution
LCM's: $(6,7) = 42$; $(6,9) = 18$; $(6,7,9) = 126$ $\lfloor 500 \div 18 \rfloor = 27$ $\lfloor 500 \div 42 \rfloor = 11$ $\lfloor 500 \div 126 \rfloor = 3$ $n = 119 - 27 - 11 + 3 = 84$
Specific behaviours
✓ divisible by $18, 42$ ✓ divisible by 126 ✓ correct number

See next page

Question 20 **(7 marks)**

$ABCD$ is a trapezium with $\overrightarrow{AB}$ parallel and in the same direction to $\overrightarrow{DC}$.

(a) Sketch a labelled diagram of $ABCD$. (1 mark)

Solution

Specific behaviours
✓ correct diagram

(b) Show that $\overrightarrow{DB} + \overrightarrow{AC} = \overrightarrow{DC} + \overrightarrow{AB}$. (2 marks)

Solution
$\overrightarrow{DB} + \overrightarrow{AC} = \left(\overrightarrow{DA} + \overrightarrow{AB}\right) + \left(\overrightarrow{AB} + \overrightarrow{BC}\right)$ $= \left(\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC}\right) + \overrightarrow{AB}$ $= \overrightarrow{DC} + \overrightarrow{AB}$
Specific behaviours
✓ splits $\overrightarrow{DB}$ and $\overrightarrow{AC}$ ✓ groups vectors that make $\overrightarrow{DC}$

(c) M lies on AC and N lies on BD so that $AM:MC = BN:ND = 2:3$. Use a vector method to prove that $ABNM$ is a trapezium. (4 marks)

Solution
$\overrightarrow{AM} = \frac{2}{5}\overrightarrow{AC}, \quad \overrightarrow{AN} = \overrightarrow{AB} + \frac{2}{5}\overrightarrow{BD}$ $\overrightarrow{NM} = \overrightarrow{AM} - \overrightarrow{AN}$ $= \frac{2}{5}\overrightarrow{AC} - \left(\overrightarrow{AB} + \frac{2}{5}\overrightarrow{BD}\right)$ $= \frac{2}{5}\left(\overrightarrow{AC} - \overrightarrow{BD}\right) - \overrightarrow{AB}$ $= \frac{2}{5}\left(\overrightarrow{DB} + \overrightarrow{AC}\right) - \overrightarrow{AB}$ $= \frac{2}{5}\left(\overrightarrow{DC} + \overrightarrow{AB}\right) - \overrightarrow{AB}$ [from (b)] But $\overrightarrow{DC} = k\overrightarrow{AB}$ $\overrightarrow{NM} = \frac{2}{5}\left(k\overrightarrow{AB} + \overrightarrow{AB}\right) - \overrightarrow{AB}$ $= \left(\frac{2k-3}{5}\right)\overrightarrow{AB} \Rightarrow ABNM$ is trapezium
Specific behaviours
✓ vectors for M and N ✓ obtains $\overrightarrow{NM}$ without M and N ✓ obtains $\overrightarrow{NM}$ in terms of $\overrightarrow{AB}, \overrightarrow{DC}$ ✓ obtains $\overrightarrow{NM}$ in terms of $\overrightarrow{AB}$

See next page

Question 21 **(8 marks)**

A helicopter, with a maximum speed through still air of 240 km/h, leaves its base at A to fly to a destination at B, where $\overrightarrow{\mathrm{AB}} = (-45\mathbf{i} + 80\mathbf{j})\ \mathrm{km}$.

There is a steady wind of velocity $(-\mathbf{i} - 3\mathbf{j})$ km/h.

The velocity that the pilot of the helicopter must set to travel from A to B is $(\mathrm{a}\mathbf{i} + \mathrm{b}\mathbf{j})$, where a and b are constants.

(a) Determine the velocity vector the helicopter pilot should set in order to fly directly from A to B in the shortest amount of time. (6 mark)

$$-45\mathrm{i} + 80\mathrm{j} = \lambda[(\mathrm{a} - 1)\mathrm{i} + (\mathrm{b} - 3)\mathrm{j}]$$

$$\begin{cases} -45 = \lambda(a - 1) \\ 80 = \lambda(b - 3) \\ 240^2 = a^2 + b^2 \end{cases}$$

$$a = -115.62, b = 210.32, \lambda = 0.386$$

(b) What is the shortest journey time, to the nearest minute? (2 marks)

$$t = 0.386 \times 60$$
$$t = 23\ minutes$$

End of questions

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Supplementary page

Question number: _____

Supplementary page

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